Polynomial Approximations and Series

I. Sequence – an infinite list of terms denoted by $\{a_n\}_{n=1}^{\infty}$

A sequence converges if $\lim_{n\to\infty} a_n$ is a real number

q.1 A sequence diverges if $\lim_{n\to\infty} a_n$ oscillates or goes to infinity

II. Series – the sum of a sequence denoted by $\sum_{n=1}^{\infty} a_n$

Consider the partial sums of a series

q.2 $S_1 = a_1$ (sum of first term)

$S_2 = a_1 + a_2$ (sum of first two terms)

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$

These partial sums form a sequence $\{S_1, S_2, S_3, \ldots, S_n, \ldots\}$

If the $\lim_{n\to\infty} S_n$ is a real number, we say the series converges.

III. Special Series

A. Geometric Series $\sum_{n=0}^{\infty} ar^n$

q.2 • each term is the previous term multified by r (common ratio)

• if $|r| < 1$ the series converges to $\frac{a}{1-r}$

Ex/

Write the decimal $.36666\overline{6}$ as a fraction.

This is equivalent to $\frac{3}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \cdots$

or $\frac{3}{10} + \sum_{n=0}^{\infty} \frac{6}{100}\left(\frac{1}{10}\right)^n = \frac{3}{10} + \frac{6/100}{1 - 1/10}$

$= \frac{3}{10} + \frac{6}{100}\cdot\frac{10}{9} = \frac{3}{10} + \frac{2}{30} = \frac{11}{30}$

B. Harmonic Series

q.3 $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

• this series diverges.

C. Alternating Series

q.5 A series in which the terms alternate signs.

Error Bound: the difference btw the total sum and the partial sum is the error. This is bounded by $|R_N| \le a_{n+1}$ (the next term).

IV. Testing for Convergence or Divergence of Series.

A. Divergence Test (n^{th} Term Test)

If $\lim_{n\to\infty} a_n$ does not approach zero, the series diverges

p.2 * Note: If the limit does approach zero, we do not know if it converges or diverges!

Ex/ $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges b/c $\lim_{n\to\infty} \frac{n}{n+1} = 1$

B. Geometric Series $\sum_{n=0}^{\infty} ar^n$

p.2 Ex/ $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$

Converges if $|r| < 1$ Converges b/c

Diverges if $|r| > 1$ $\pi/6 < 1$

C. Telescoping Series

p.2 A series in which all the terms after the first few cancel out the limit of the n^{th} term approaches zero.

Ex/ $$\sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) = \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{11}}\right) + \cdots = \frac{1}{3}$$

$\therefore$ Converges to $1/3$

D. P-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Ex/ $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ Diverges b/c $1/2 \le 1$

p.3 Converges if $p > 1$

Diverges if $p \le 1$

*Note: The harmonic series is a special case where $p = 1$.

E. Integral Test $\sum_{n=1}^{\infty} a_n$

p.3

Let $f(x)$ be the function formed by replacing n with x. If $\int_1^{\infty} f(x)\,dx$ converges to a real #, the series converges. If $\int_1^{\infty} f(x)\,dx$ approaches ∞, the series diverges

Ex/ $\sum_{n=1}^{\infty} ne^{-n^2} \Rightarrow \int_1^{\infty} xe^{-x^2}\,dx$ $u = -x^2$, $du = -2x\,dx$ $\therefore$ Converges

$$\Rightarrow \lim_{T\to\infty} -\frac{1}{2}\int_1^{T} e^u\,du = \lim_{T\to\infty} -\frac{1}{2}e^{-x^2}\Big]_1^T = -\frac{1}{2}\left(\cancelto{0}{\frac{1}{e^{T^2}}} - \frac{1}{e}\right) = \frac{1}{2e}$$

F. Alternating Series $\sum_{n=1}^{\infty} (-1)^n a_n$

A series in which the signs change (or alternate)

9.5

Converges if $a_{n+1} \le a_n$ and $\lim_{n\to\infty} a_n = 0$

An alternating series converges conditionally if the absolute value of the series diverges.

An alternating series converges absolutely if the absolute value of the series converges.

Ex/ ① $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ Since $\lim_{n\to\infty} \frac{3}{4n+1} = 0$ and $\frac{3}{4(n+1)+1} \le \frac{3}{4n+1}$ $\Big\}$ thus Converges

② $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ Clearly alternates but the terms are not necessarily decreasing.

Consider $\left|\frac{\cos k}{k^2}\right| \le \left|\frac{1}{k^2}\right|$ a convergent p-series.

Since the series converges absolutely, the series converges!

G. Ratio Test

9.6

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series converges

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, the series diverges.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, inconclusive.

Ex/ $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ $\lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n\to\infty} \frac{n+1}{10} \Rightarrow \infty$ ∴ Diverges.

H. Root Test

9.6

If $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$, the series converges

If $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$, the series diverges

If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, inconclusive

Ex/ $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$ $\lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{2n+1}\right)^n} = \lim_{n\to\infty} \left(\frac{n+1}{2n+1}\right) = \frac{1}{2}$ ∴ Converges

I. Direct Comparison Test

Compare two series $\sum a_n$ and $\sum b_n$ where $a_n \leq b_n \ \forall n$

9.4 If $\sum b_n$ converges then a_n also converges

If $\sum a_n$ diverges then b_n also diverges

Ex/ $\sum_{n=1}^{\infty} \frac{1}{3n-1}$ compare to $\sum \frac{1}{3n}$ $\quad \frac{1}{3n-1} > \frac{1}{3n}$ (the harmonic series

$\therefore \sum \frac{1}{3n-1}$ diverges

J. Limit Comparison Test

9.4 Compare the series $\sum a_n$ with a known series $\sum b_n$

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, a positive real number

then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

Ex/ $\sum_{n=1}^{\infty} \frac{3}{4n+1}$ compare to $\sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{3}{4n+1}}{1/n} = \lim_{n\to\infty} \frac{3n}{4n+1} = \frac{3}{4} \quad \therefore \text{ Both diverge.}$$

V. Taylor Series – a polynomial series used to represent a function.

9.6/9.10 A. $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^n(c)}{n!}(x-c)^n + \cdots$

is the power series centered at c.

Ex/

$f(x) = \sin x$ at $x = \pi/2$ $\quad f(\pi/2) = 1$

$f'(x) = \cos x \qquad f'(\pi/2) = 0$

$f''(x) = -\sin x \qquad f''(\pi/2) = -1$

$f'''(x) = -\cos x \qquad f'''(\pi/2) = 0$

$f^{iv}(x) = \sin x \qquad f^{iv}(\pi/2) = 1$

$f^{v}(x) = \cos x \qquad f^{v}(\pi/2) = 0$

$f^{vi}(x) = -\sin x \qquad f^{vi}(\pi/2) = -1$

thus the Taylor series is

$$f(x) = f(\pi/2) + f'(\pi/2)(x-\pi/2) + \frac{f''(\pi/2)}{2!}(x-\pi/2)^2 + \frac{f'''(\pi/2)}{3!}(x-\pi/2)^3 + \cdots$$

$$= 1 + 0 + \frac{-1}{2!}(x-\pi/2)^2 + 0 + \frac{1}{4!}(x-\pi/2)^4 + \cdots$$

$$= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \cdots$$

B. A <u>Maclaurin Series</u> is simply a Taylor series centered at $c=0$.

9.7/9.10 A few special Maclaurin series include:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{x^n}{n!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

C. <u>Finding series from known series.</u>

Ex/ ① Find the series for $\cos\sqrt{x}$ (Use series for $\cos x$ and replace x with $\sqrt{x}$)

9.9/9.10

yields $\boxed{\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{x^{n/2}}{n!} + \cdots}$

② Find a series for $g(x) = \arctan x$

a) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$

thus $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots + x^n + \cdots$

and $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots + x^{2n} + \cdots$

Since $\arctan x = \int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + x^8 + \cdots + x^{2n} + \cdots$

$\boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots}$

D. <u>Radius and Interval of Convergence</u>

Uses the ratio test to determine when the series converges

9.7 Ex/ $\sum_{n=0}^{\infty} (-1)^{n+1} n x^n$

$$\lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{n \cdot x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x}{n}\right| = |x| < 1 \text{ then converges}$$

thus radius of convergence $\boxed{R=1}$ and interval is $\boxed{-1<x<1}$

check the endpoints

if $x=-1$ $\sum_{n=0}^{\infty} -n$ diverges if $x=1$ $\sum(-1)^{n+1} n$ diverges

Convergence of a power series has 3 possibilities:

a) Converges only at the center c

b) Converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$ for some positive real number R.

c) Converges absolutely for all x.

* Note: the endpts of the interval must each be tested individually for convergence or divergence.

E. Lagrange Error Bound.

For the n^{th} Taylor polynomial the error is less than

9.7 $$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

Ex/ $e^1 \approx 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!}$ $\quad f(x) = e^x$

This is the 5^{th} Maclaurin polynomial thus $R_5(x) = \frac{(e^z)^{(6)}}{(n+1)!} \cdot x^6$

$$R_5(x) \leq \frac{e^1}{6!} \cdot 1^6 \approx \underline{\underline{.00378}}$$